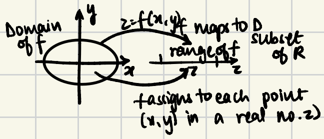
15.1

Graphs and level curves



Level curve: take a plane across: the curve that cuts the plane

15.2

limits and continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L$$

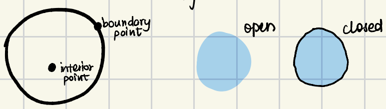
$$|f(x,y) - L| < \varepsilon$$

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

$f(x,y) = c \quad \lim_{(x,y)\to(a,b)} c = c$

$f(x,y) = x \quad \lim_{(x,y)\to(a,b)} x = a$

$f(x,y) = y \quad \lim_{(x,y)\to(a,b)} y = b$



THEOREM 15.2 Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is a constant, and $n > 0$ is an integer.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$
7. **Root** $\lim_{(x,y)\to(a,b)} (f(x,y))^{1/n} = L^{1/n}$, where we assume $L > 0$ if n is even.

← Textbook

15.3

Partial Derivatives

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$ for x

≡ similarly for y

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

Notation: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$

Clairaut Equality $f_{xy} = f_{yx}$ (if continuous)

differentiability

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$
$$= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

→ implies continuous

1. value defined
2. limit exists
3. value = limit

Lecture 4

Differentiability

Explicit: differentiable ⇒ continuous

Contrapositive: assume not differentiable
→ at least one of the partial derivatives of f_x and f_y not continuous at (a,b)
OR there is no (a,b) (partial derivative defined at all points of set)

not continuous ⇒ not differentiable

15.4 Chain Rule

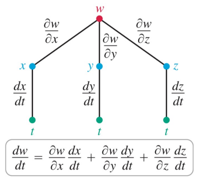


Figure 15.37

$\frac{d}{dt}$ ⇒ no branching

$\frac{\partial}{\partial t}$ ⇒ branching

← Textbook

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Implicit Differentiation $\frac{dy}{dx} = -\frac{F_x}{F_y}$ → $\neq 0$ $\quad F(x,y)$

15.5 Directional Derivatives and the Gradient

slope (secant) $= \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

⇓

(tangent) $= \lim_{h\to 0}$ " " "

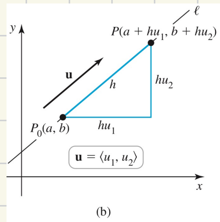


Figure 15.46

← Textbook

Directional derivative

$$D_u f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$

$$D_{u_1} f_x(a,b) = f_x(a)(u_1) + f_x(b)(u_1)$$

$$D_{u_2} f_y(a,b) = f_y(a)(u_2) + f_y(b)(u_2)$$

Gradient (2D) take every term with x in it, partial diff. wrt x, put values ⇒ gradient ⇓ multiply by directional u = directional derivative

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = f_x(x,y)\,i + f_y(x,y)\,j$$

$$D_u f(a,b) = \nabla f(a,b) \cdot u$$

but when direction given? ⇒ make it 'u' then multiply by gradient = directional derivative

$$D_u f(a,b) = \nabla f(a,b) \cdot u$$
$$= |\nabla f(a,b)| \cos\theta$$

$D_u = u \times \nabla f(x,y,z) = f_x(x,y,z)\,i\; u_1$
$+ f_y(x,y,z)\,j\; u_2$
$+ f_z(x,y,z)\,k\; u_3$

THEOREM 15.11 Directions of Change
Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

← Textbook

$$\frac{d}{dt} f(x,y) = \nabla f(x,y) \cdot r'(t)$$

15.6 Tangent Planes and Linear Approximation

Implicit Functions $\quad F(x,y,z) = 0$

$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$ → equation of the plane

$F(x,y) = z$ $\quad$ Explicit Functions $\quad z = L(x,y)$

Linear Approx. $\quad L_f(x,y)$ at (a,b)

$\| $

$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$ →

OR $= f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + \cancel{f(a,b,c)}$ for "change" Δz or differential dz

$L_f(\vec{x}_0) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0)$ "I like this one" - Satti

Lecture 6

same

lec 8

Small Signal Modeling

Jacobian $J(u,v) = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}$ variation $\frac{\partial(x,y)}{\partial(u,v)}$

$y = g(x, u)$

y ↓ output, x ↓ state, u ↓ input

$u \to \boxed{x} \to y$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = J_f \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + B_f \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

↓

small = variable

everything else = capital const

$$\begin{bmatrix} \left.\frac{\partial f_1(x_1 x_2 U_1 U_2)}{\partial x_1}\right|_{x_1 = X_1} & \left.\frac{\partial f_1(X_1 x_2 U_1 U_2)}{\partial x_2}\right|_{x_2 = X_2} \\ \left.\frac{\partial f_2(x_1 x_2 U_1 U_2)}{\partial x_1}\right|_{x_1 = X_1} & \left.\frac{\partial f_2(x_1 x_2 U_1 U_2)}{\partial x_2}\right|_{x_2 = X_2} \end{bmatrix}$$

→ f

↓ x

given $\frac{dx_1}{dt}$, $\frac{dx_2}{dt}$, U_1, U_2

x_1, x_2, U_1, U_2 → all constants no variables

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

equilibrium: value of Y_1 Y_2 within constraints

small perturbations

$u_1(t) = U_1 + \hat{u}_1(t)$

lec 9

Double integrals over rectangular regions

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Lec 10

Double integrals over non-rectangular regions

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

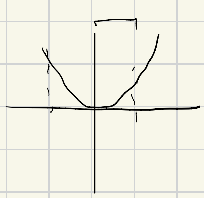
Volume between bounding surfaces

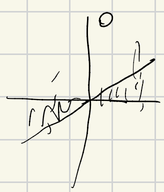
$$V = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$$

Conditional symmetries ← Prof Dawson

1. If $(f(x,y) = \overset{\text{even}}{f(x,-y)})$ ***AND*** $(f(x,y) = \overset{\text{even}}{f(-x,y)}) \rightarrow (f(x,y) = f(-x,-y))$ then
   $\int_{-b}^{b}\int_{-a}^{a} f(x,y)dxdy = 4\int_0^b\int_0^a f(x,y)dxdy$

2. If $(f(x,y) = -f(x,\overset{\text{odd}}{-y}))$ ***OR*** $(\overset{\text{odd / even}}{f(x,y)} = -f(-x,y))$ then
   $\int_0^b\int_{-a}^{a}(f(x,y) + f(x,-y))dxdy = 0$ or $\int_{-b}^{b}\int_0^a(f(x,y) + f(-x,y))dxdy = 0$





Lecture 11

Triple integrals (Reading 3 short)

finding limits

Theorem Triple Integrals

Let f be continuous over the region

$$D = \{(x,y,z): a \le x \le b, g(x) \le y \le h(x), G(x,y) \le z \le H(x,y)\}$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x,y,z)dV = \underset{0}{\int_a^b} \underset{1}{\int_{g(x)}^{h(x)}} \underset{2}{\int_{G(x,y)}^{H(x,y)}} f(x,y,z)dzdydx$$

Note: The Theorem is a version of Fubini's Theorem. Five other versions could be written for the other orders of integration.

← Prof Dawson

Changing Order of Integration

→ graph → midpoints ?

Average Value of a function of Three variables

Definition Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of f over D is

$$\bar{f} = \frac{1}{\text{volume of } D} \iiint_D f(x,y,z)dV$$

← Prof Dawson

$x \times y \times z$ = volume

Change of Variables in multiple integrals (2D and 3D)

$f(x,y) \rightarrow \iint f \, dx\,dy \rightarrow \iint du\,dv$

$x = g(u,v)$

$y = h(u,v)$

$$\Rightarrow \iint_{x=a(y)}^{x=b(y)} f(g(u,v), h(u,v)) \left| J \right|_{det} du\,dv$$

$$J(x,y) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}_{det}$$

Lec 14

∬ in polar

T	T^{-1}	Bounds
$x = r\cos\theta$	$r^2 = x^2+y^2$	$r \geq 0$
$y = r\sin\theta$	$\tan\theta = \frac{y}{x}$	$\theta \in [0, 2\pi]$

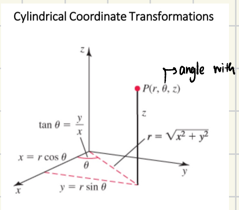
$$\iint_R f(x,y)\,dx\,dy = \iint_R f \cdot \underbrace{r}_{|J|}\,dr\,d\theta$$

$$\int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Average: $\dfrac{\iint f\, r\,dr\,d\theta}{\iint r\,dr\,d\theta}$ } area

Lec 15

∭ in cylindrical coordinates



angle with +ve x
thumbed from z

T	T^{-1}	Bound
$x = r\cos\theta$	$r^2 = x^2+y^2$	$r \geq 0$
$y = r\sin\theta$	$\tan\theta = \frac{y}{x}$	$\theta \in [0, 2\pi]$
$z = z$	$z = z$	$z \in \mathbb{R}$

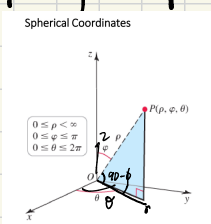
$$\iiint f(x,y,z)\,dx\,dy\,dz = \int_\beta^\alpha \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f \cdot r\, dz\,dr\,d\theta$$

r, θ, z → $\rho^2\sin\phi$
$\rho = r^2 + z^2$
θ
$\phi = \tan^{-1}\left(\frac{r}{z}\right)$

ρ, θ, ϕ → $\rho^2\sin\phi$
$r = \rho^2 - \left(\frac{r}{\tan\phi}\right)^2$
θ
$z = \frac{r}{\tan\phi}$

Lec 16

Triple integrals in spherical coordinates



T	T^{-1}	Bounds
$x = r\sin\phi\cos\theta$	$r^2 = x^2+y^2+z^2$	$r \geq 0$
$y = r\sin\phi\sin\theta$	$\tan\theta = \frac{y}{x}$	$\theta \in [0, 2\pi]$ (wh)
$z = r\cos\phi$	$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$	$\phi \in [0, \pi]$ (not strict)
for Jacobian	for Boundary	

$\frac{r}{\rho} = \sin(90-\phi) = \cos\phi$

$\rho = \frac{r}{\cos\phi}$

$r = \rho\cos\phi$

$a \leq \phi \leq b \qquad b - a \leq \pi$

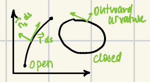
$$\int_\alpha^\beta \int_a^b \int_{g(\phi,\theta)}^{h(\phi,\theta)} f \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Lec 17

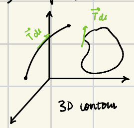
Line integrals of vector dot product

Types of Integrals

Contour wire (curve) in 2D or 3D) → open, closed ✓



2D contour



3D contour

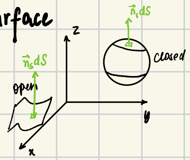
no "normal" ⇒ line integral

open: $\int_C \vec{F} \cdot \vec{T}$ closed: $\oint \vec{F} \cdot \underbrace{\vec{T}}_{\text{circulation}} ds$ } 2D, along a line

circulation/m ⇒ scalar

Circulation density

Surface



no "tangent"

flux → vector

$\int_C \vec{F} \cdot \vec{n}_s ds$ $\oint \vec{F} \cdot \vec{n}_s ds$ } 2D, along a line

F/m (flux density)

open: $\iint \vec{F} \cdot \vec{n}_s dS$ close: $\oiint \vec{F} \cdot \vec{n}_s dS$

have to be continuous
don't have to be diffentiable
eg. cube also valid surface

} in 3D, over a surface

flux/m^2

Special Cases:

1) 2D if $\vec{F} = \lambda \vec{T}$ (λ: line density, scalar)

$\int_C \vec{F} \cdot \vec{T} ds = \int_C \lambda ds$

2) $\vec{F} = \sigma \vec{n}_s$ (σ: surface density)

$\iint_S \vec{F} \cdot \vec{n}_s dS = \iint_S \sigma dS$

Average mass $= \frac{\int \lambda ds}{\int ds} \Rightarrow \frac{\iint \sigma dS}{\iint dS}$

Parameterized Explicit

$\langle \dot{x}, -\dot{y} \rangle$
or
$\langle -\dot{x}, \dot{y} \rangle$

$T = \frac{\dot{r}}{|\dot{r}|}$

$n_s = \frac{\langle -\dot{y}, -\dot{x} \rangle}{|\dot{r}|}$

2D parameterized $x(t)$ $y(t)$

$\vec{r} = \langle x, y \rangle$ $\dot{\vec{r}} = \langle \dot{x}, \dot{y} \rangle$

$ds = \sqrt{\dot{x}^2 + \dot{y}^2}$ $T ds = \dot{\vec{r}} dt = \langle \dot{x}, \dot{y} \rangle dt$

right handed system $n_s ds = \langle \dot{y}, -\dot{x} \rangle dt$

Circulation: $\oint \vec{F} \cdot \vec{T} ds = \int_{t_a}^{t_b} \langle f, g \rangle \cdot \langle \dot{x}, \dot{y} \rangle dt$

flux → $\oint \vec{F} \cdot \vec{n}_s ds = \int_{t_a}^{t_b} \langle f, g \rangle \cdot \langle \dot{y}, \dot{x} \rangle dt$

2D vector field

$\langle f(x,y), g(x,y) \rangle =$

$\vec{n}_s$ in 3D

$\vec{r} = \langle x, y, z \rangle$ $\dot{\vec{r}} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$

$x = g(u,v)$ $y = h(u,v)$ $z = f(u,v)$

$\vec{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

$\vec{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

$dS = \underbrace{t_u \times t_v \, du \, dv}_{\text{jacobian}}$

$$\begin{array}{c|ccc} & x & y & z \\ \hline u & & & \\ v & & & \end{array}$$

$\vec{n}_s = \frac{\vec{t}_u \times \vec{t}_v}{|\vec{t}_u \times \vec{t}_v|}$ $\vec{n}_s dS = \vec{t}_u \times \vec{t}_v \, du \, dv$

$dS = |\vec{t}_u \times \vec{t}_v| \, du \, dv$

$T \Rightarrow$ tangent : $\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle \Rightarrow$ $\frac{\dot{r}}{|\dot{r}|}$

$\hat{T}\, ds \Rightarrow \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle$ $\langle |\dot{r}| \rangle$ $\dot{r}\, dt$

$\langle |\dot{r}| \rangle$

$r = \langle x, y, z \rangle$

$\dot{r} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$

$n_s\, ds$

$n \Rightarrow \dfrac{\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x} \rangle}{|\dot{r}|} \rightarrow$ 3D $\Rightarrow t_u \times t_v\, du dv \Rightarrow \langle \frac{\partial r}{\partial x} \rangle \times \langle \frac{\partial r}{\partial y} \rangle\, du dv$

$u \quad v$

$t_x \times t_y$

$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}$

circulation $\int \vec{F} \cdot \vec{T} ds$

flux $\int \vec{F} \cdot \vec{n} ds$

Circulation $= \iint_{y\,x}$ curl $n\, dx dy$

flux $= \iint_{y\,x}$ divergence $dx dy$

Divergence $\nabla \cdot F$

Curl $\nabla \times F$

$\nabla = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

Divergence theorem

$$\oint F \cdot n ds = \iint \rho\, dA$$

$$\oiint F \cdot n ds = \iiint \rho\, dV$$

$\rho = \nabla \cdot F$

Stoke's theorem

$$\oint F \cdot T ds = \iint \nabla \times F\, \hat{n} ds$$

$\frac{d}{d\theta} \sin\theta \rightarrow \cos\theta$ $\quad \int \cos\theta = \sin\theta$

$\frac{d}{d\theta} \cos\theta \rightarrow -\sin\theta$ $\quad \int \sin\theta = -\cos\theta$

$4\pi r^2 f(r)$

Shell: $\sigma\, \delta(r-R)$ $\quad (r^2 \sin\phi) \Rightarrow 4\pi R^2 (\sigma)$

Point: $\frac{Q}{2\pi r^2}\, \delta(r) \Rightarrow Q$

Sphere : ρ $\quad (r^3 \sin\phi) \Rightarrow \frac{4}{3}\pi r^3 (\rho_0)$

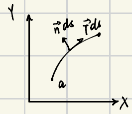
$2\pi r\, f(r)$

Cylinder : ρ : $\rho\, \pi (R_2^2 - R_1^2)$

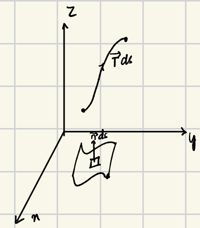
shell : $J_s\, \delta(r-R)$: $2\pi R \quad J_s$

filament : $\frac{I\, \delta(r)}{\pi r}$: I

line and surface integrals



line in 3D



circulation

line integral (2D, 3D)

lines/contours

$$\int_c \vec{F}\cdot\vec{T}\,ds \qquad \oint_c \vec{F}\cdot\vec{T}\,ds$$

scalar (λ line density)

$$\oint \text{ or } \int_c \lambda\,ds = \text{integral along the contour}$$

flux (2D → bullshit 🎀)

surface integral (3D)

2D: $\int \vec{F}\cdot\vec{n}\,ds$ 3D: $\iint \vec{F}\cdot\vec{n}\,dS$

$\vec{r} = \langle x, y, z\rangle$

$\vec{T} = \dfrac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$

vector valued function

$\vec{T}ds = \dot{\vec{r}}\,dt$

$ds = |\dot{\vec{r}}|\,dt$

scalar function

$\vec{n}_s = \langle \dot{y}, -\dot{x}\rangle$ right handed system

or $\langle -\dot{y}, \dot{x}\rangle$ left handed system

• 3D

Circulation and flux in 2D

parametric $a \le t \le b$

$\vec{r} = \langle x(t), y(t), z(t)\rangle$

explicit → one of the variables has to be independent

$\vec{r} = \langle x, y(x), z(x,y)\rangle$

Circulation and flux in 3D →

circulation

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{\langle \dot{x}, \dot{y}\rangle}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$\dot{z}$, $+\dot{z}^2$

$\vec{T}ds = \langle \dot{x}, \dot{y}\rangle dt$

$\dot{\vec{r}}dt$ in general

flux 2D

arclength

$$ds = |\dot{\vec{r}}|\,dt = \sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$\vec{n} = \frac{\langle \dot{y}, \dot{x}\rangle}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$\vec{n}\,ds = \langle -\dot{y}, \dot{x}\rangle dt$

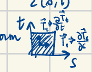
↪ does not hold for 3D

3D $\vec{r} = (x, y, z)$

$$\oiint_S \underbrace{\vec{F}(x,y,z)}\cdot\vec{n}_s\,dS$$

$x(s,t)$
$y(s,t)$
$z(s,t)$

transform



$$\vec{T}_s = \frac{\partial \vec{r}}{\partial s} = \left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right\rangle$$

$$\vec{T}_t = \frac{\partial \vec{r}}{\partial t} = \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right\rangle$$

$$\vec{n}_s\,dS = (\vec{T}_s \times \vec{T}_t)\,ds\,dt$$

$$\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\,ds\,dt$$

quick tip
how to do cross product

$$\vec{T}_s \times \vec{T}_t = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}$$

Surface integral

$$\iint_S \lambda\,|\vec{T}_s \times \vec{T}_t|$$

vector

$$\iint_{v\,u} \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot (\vec{t}_u \times \vec{t}_v)\,du\,dv$$

scalar

$$\iint_{v\,u} f(x(u,v), y(u,v), z(u,v)) \cdot |\vec{t}_u \times \vec{t}_v|\,du\,dv$$

} parameterized form

vector

$$\iint_{x\,y} \vec{F}(x, y, z(x,y)) \cdot (\vec{t}_x \times \vec{t}_y)\,dx\,dy$$

scalar

$$\iint_{x\,y} f(x, y, z(x,y)) \cdot |\vec{t}_x \times \vec{t}_y|\,dx\,dy$$

} explicit form

explicit representation

surface $\langle x, y, z(x,y)\rangle$

reimann $Q = \iint_S \sigma(x, y, z(x,y))\,\underset{\sim}{dS}$

$|t_x \times t_y|\,dx\,dy$

flux $\iint_S F\cdot dS$

Summary: Computing Surface Area Integrals

	Parameterized representation	Explicit representation
Position vector	$\boldsymbol{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$ bounds on u and v	$\boldsymbol{r} = \langle x, y, z(x,y)\rangle$ bounds on x and y
Vectors defining the tangential plane	$\boldsymbol{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle$ $\boldsymbol{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$	$\boldsymbol{t}_x = \left\langle 1,0,\frac{\partial z}{\partial x}\right\rangle$ $\boldsymbol{t}_y = \left\langle 0,1,\frac{\partial z}{\partial y}\right\rangle$
Unit vector $\boldsymbol{n}_S$ normal to the tangential plane	$\frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{\lvert\boldsymbol{t}_u \times \boldsymbol{t}_v\rvert}$	$\frac{\boldsymbol{t}_x \times \boldsymbol{t}_y}{\lvert\boldsymbol{t}_x \times \boldsymbol{t}_y\rvert} = \frac{\langle -z_x, -z_y, 1\rangle}{\sqrt{1+(z_x)^2+(z_y)^2}}$
dS (magnitude of differential area)	$\lvert\boldsymbol{t}_u \times \boldsymbol{t}_v\rvert du\,dv$	$\sqrt{1+(z_x)^2+(z_y)^2}\,dy\,dx$
$\boldsymbol{dS} = \boldsymbol{n}_S dS$	$\boldsymbol{t}_u \times \boldsymbol{t}_v du\,dv$	$\langle -z_x, -z_y, 1\rangle dy\,dx$

Note: there are two options for $\boldsymbol{n}_S$: pointing upwards or pointing downwards. The problem statement or convention will dictate which version to adopt.

	Parameterized representation	Explicit representation
Tangential plane differential vectors	$\boldsymbol{t}_r = \left\langle \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right\rangle$ $\boldsymbol{t}_\theta = \left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right\rangle$ $\boldsymbol{t}_r = \langle \cos\theta, \sin\theta, 1\rangle$ $\boldsymbol{t}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$	$\boldsymbol{t}_x = \left\langle 1,0,\frac{\partial z}{\partial x}\right\rangle$ $\boldsymbol{t}_y = \left\langle 0,1,\frac{\partial z}{\partial y}\right\rangle$ $\boldsymbol{t}_x = \left\langle 1,0,\frac{x}{\sqrt{x^2+y^2}}\right\rangle$ $\boldsymbol{t}_y = \left\langle 0,1,\frac{y}{\sqrt{x^2+y^2}}\right\rangle$
Unit vector $\boldsymbol{n}_S$ normal to the tangential plane	$\frac{\boldsymbol{t}_r \times \boldsymbol{t}_\theta}{\lvert\boldsymbol{t}_r \times \boldsymbol{t}_\theta\rvert} = \frac{1}{r\sqrt{2}}\left\lVert\begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0\end{matrix}\right\rVert_{det}$ $= \frac{1}{\sqrt{2}}\langle -\sin\theta, -\cos\theta, 1\rangle$	$\frac{\boldsymbol{t}_x \times \boldsymbol{t}_y}{\lvert\boldsymbol{t}_x \times \boldsymbol{t}_y\rvert} = \frac{\langle -z_x, -z_y, 1\rangle}{\sqrt{1+(z_x)^2+(z_y)^2}}$ $= \frac{1}{\sqrt{2}}\left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle$
dS (magnitude)	$\lvert\boldsymbol{t}_r \times \boldsymbol{t}_\theta\rvert dr\,d\theta = \sqrt{2}r\,dr\,d\theta$	$\sqrt{1+(z_x)^2+(z_y)^2}\,dy\,dx = \sqrt{2}\,dy\,dx$
$\boldsymbol{dS} = \boldsymbol{n}_S dS$	$\boldsymbol{t}_r \times \boldsymbol{t}_\theta dr\,d\theta = \langle -r\sin\theta, -r\cos\theta, r\rangle dr\,d\theta$	$\langle -z_x, -z_y, 1\rangle dy\,dx$ $= \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle dy\,dx$

← Prof Dawson

Summary: Computing Surface Area Integrals

Parameterized representation	Explicit representation	Meaning
$\iint_S f(x(u,v), y(u,v), z(u,v)\ \lvert\boldsymbol{t}_u \times \boldsymbol{t}_v\rvert du\,dv$	$\iint_R f(x,y,z(x,y))\sqrt{1+(z_x)^2+(z_y)^2}\,dy\,dx$	Physical/geometric
$\iint_S \boldsymbol{F}(x(u,v), y(u,v), z(u,v)) \cdot \boldsymbol{t}_u \times \boldsymbol{t}_v du\,dv$	$\iint_A \boldsymbol{F}(x,y,z(x,y)) \cdot \langle -z_x, -z_y, 1\rangle dy\,dx$	Flux

$\nabla\cdot$ **Divergence** $\nabla\cdot F_{circ} = 0$

units:

2D $\lim_{A\to 0} \frac{\oint_c F\cdot n_s ds}{A}$ $(flux/m)$ $\nabla\cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle F_x, F_y \rangle$ $(flux/m^2)$ scalar output

3D $\lim_{V\to 0} \frac{\oiint_s F\cdot n_s ds}{V}$ $(flux/m^2)$ $\nabla\cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_x, F_y, F_z \rangle$ $(flux/m^3)$ (dot product)

$\nabla\times$ **Curl** $\nabla\times F_{flux} = 0$

2D $\lim_{A_s\to 0} \frac{\oint_c F\cdot T ds}{A_s}$ $(circ/m)$ $\nabla\times F = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ $(circ/m^2)$ vector output (cross product)

3D $\nabla\times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}_{det}$ $(circ/m^3)$

Lagrange's formula (vector Laplacian)

$\nabla(\nabla\cdot A) - \nabla\times(\nabla\times A) = \nabla^2 A$

" Green's Theorem " ↓ 2D version

$\oiint_s \vec{F}\cdot\vec{n}_s dS = \iiint_v \nabla\cdot\vec{F} dV$

gives net flux

Helmholtz Decomposition Theorem

$F = \underbrace{-\nabla\phi}_{F_{flux}} + \underbrace{\nabla\times A}_{F_{circ}}$

scalar potential vector potential

$\nabla\times\vec{F} = \nabla\times\nabla\times\vec{A} = \vec{J}$ → 0 source free → Stoke's theorem $\iint_s \vec{J}\cdot d\vec{S} = I_{enclosed}$

$\vec{J}$ rotation

$\nabla\cdot\vec{F} = -\nabla^2\phi = \rho$ → 0 irrotational → divergence theorem $\iiint_v \rho dV = Q_{enclosed}$

$+\rho$ $-\rho$

Stoke's (curl) **Theorem**

$\oint_c \vec{F}\cdot\vec{T} ds = \iint \nabla\times\vec{F} (\vec{n}_s ds)$ → $(t_u \times t_v) du dv$

gives net flux

Dirac Distributions

ρ filament charge Density $\delta(r)$

surface charge density $\delta(r-R)$

Point $Q = \iiint_v \rho\,\delta(x)\delta(y)\delta(z)\, dz\,dy\,dx$ $Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{r>0} \frac{Q}{4\pi r'^2}\delta(r') r'^2 \sin\phi\, dr' d\phi\, d\theta$

Surface $Q = \iiint_v \sigma\,\delta(z)\, dz\,dy\,dx$ $Q = \int_0^{z}\int_0^{2\pi}\int_0^{r>R} \sigma\,\delta(r'-R) r'\, dr'\, d\theta\, dz$ $Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{r>R} \sigma\,\delta(r'-R) r'^2 \sin\phi\, dr'\, d\phi\, d\theta$

Line $Q = \iiint_v \lambda\,\delta(x)\delta(y)\, dx\,dy\,dz$ $Q = \int_0^{z}\int_0^{2\pi}\int_0^{r>0} \frac{\lambda}{\pi r'}\delta(r') r'\, dr'\, d\theta\, dz$

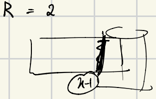
$\vec{J}$ Choose parallel $\vec{J}$ to $d\vec{s}$

Surface $I = \int_0^{y}\int_{z<0}^{z>0} J_s\,\delta(z')\hat{x} \cdot dz'\,dy'\hat{x}$ (planar) $I = \int_0^{2\pi}\int_0^{r>R} J_s\,\delta(r'-R)\hat{z} \cdot r' dr' d\theta\, \hat{z}$ (axial) $I = \int_0^{z}\int_0^{r>R} J_s\,\delta(r'-R)\hat{\theta} \cdot dr'\, dz\, d\hat{\theta}$ (circumferential)

Line $I = \int_{x<0}^{x>0}\int_{y<0}^{y>0} I\,\delta(y')\delta(x')\hat{z} \cdot dy'\,dx'\hat{z}$ $I = \int_0^{2\pi}\int_0^{r>0} \frac{I}{\pi r'}\delta(r')\hat{z} \cdot r' dr' d\theta\hat{z}$

$r' = \sqrt{x^2_{-3} + z^2_{0}} = (x-3)$

$R = 2$



Vector Transformation

2D $\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix}$ inverse/transpose $= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3D $\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \\ V_z \end{bmatrix}$ inverse/transpose $= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ right? check

$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\phi \\ V_\theta \end{bmatrix}$ inverse/transpose $= \begin{bmatrix} & & \\ & & \end{bmatrix}$ right?

Unit Vectors (relationships)

Relationship Between Unit Vectors in Different Coordinate Systems

Polar Coordinates (2d)

$r = \sqrt{x^2 + y^2}$ $\quad tan\theta = \frac{y}{x}$ $\quad \hat{r} = cos\theta\hat{x} + sin\theta\hat{y}$ $\quad \hat{\theta} = -sin\theta\hat{x} + cos\theta\hat{y}$

$\hat{r} = \frac{x}{\sqrt{x^2 + y^2}}\hat{x} + \frac{y}{\sqrt{x^2 + y^2}}\hat{y}$ $\quad \hat{\theta} = -\frac{y}{\sqrt{x^2 + y^2}}\hat{x} + \frac{x}{\sqrt{x^2 + y^2}}\hat{y}$

Cylindrical Coordinates (3d)

$r = \sqrt{x^2 + y^2}$ $\quad \tan\theta = \frac{y}{x}$ $\quad z = z$ $\quad \hat{r} = cos\theta\hat{x} + sin\theta\hat{y}$ $\quad \hat{\theta} = -sin\theta\hat{x} + cos\theta\hat{y}$ $\quad \hat{z} = \hat{z}$

$\hat{r} = \frac{x}{\sqrt{x^2 + y^2}}\hat{x} + \frac{y}{\sqrt{x^2 + y^2}}\hat{y}$ $\quad \hat{\theta} = -\frac{y}{\sqrt{x^2 + y^2}}\hat{x} + \frac{x}{\sqrt{x^2 + y^2}}\hat{y}$ $\quad \hat{z} = \hat{z}$

Spherical Coordinates (3d)

$\rho = \sqrt{x^2 + y^2 + z^2}$ $\quad \sin\varphi = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$ $\quad tan\,\theta = \frac{y}{x}$

$\hat{\rho} = cos\theta sin\varphi\hat{x} + sin\theta sin\varphi\hat{y} + cos\varphi\hat{z}$ $\quad \hat{\varphi} = cos\theta cos\varphi\hat{x} + sin\theta cos\varphi\hat{y} - sin\varphi\hat{z}$ $\quad \hat{\theta} = -sin\theta\hat{x} + cos\theta\hat{y} - 0\hat{z}$

$$\hat{\rho} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\hat{x} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\hat{y} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\hat{z}$$

$$\hat{\varphi} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}\frac{x}{\sqrt{x^2 + y^2}}\hat{x} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\frac{y}{\sqrt{x^2 + y^2}}\hat{y} - \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}\hat{z}$$

$$\hat{\theta} = -\frac{y}{\sqrt{x^2 + y^2}}\hat{x} + \frac{x}{\sqrt{x^2 + y^2}}\hat{y} - 0\hat{z}$$

← Prof Dawson's Notes

Differential Geometric Entities in Other Coordinates

This is a course in differential calculus so we are interested in the following differential geometric elements:

- differential length (a vector $\boldsymbol{ds}$ in $2d$ and $3d$) ,
- differential area, (a scalar dA) in $2d$, but an oriented differential area (a vector $\boldsymbol{dS}$) in $3d$,
- differential volume (a scalar dV) in $3d$.

	$\boldsymbol{ds}$	dA	$\boldsymbol{dS}$	dV
cartesian (2d)	$dx\hat{x} + dy\hat{y}$	$dxdy$	n.a	n.a
cartesian (3d)	$dx\hat{x} + dy\hat{y} + dz\hat{z}$	n.a	$dS_z = dxdy\hat{z}$ $dS_x = dydz\hat{x}$ $dS_y = dzdx\hat{y}$	$dzdydx$
polar (2d)	$dr\hat{r} + rd\theta\hat{\theta}$	$rdrd\theta$	n.a	n.a
cylindrical (3d)	$dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z}$	n.a	$dS_r = rd\theta dz\hat{r}$ $dS_\theta = drdz\hat{\theta}$ $dS_z = rdrd\theta\hat{z}$	$rdrd\theta dz$
spherical (3d)	$d\rho\hat{\rho} + \rho d\varphi\hat{\varphi} + \rho sin\varphi d\theta\hat{\theta}$	n.a	$dS_\rho = \rho^2 sin\varphi d\theta d\varphi\hat{\rho}$ $dS_\varphi = \rho sin\varphi d\rho d\theta\hat{\varphi}$ $dS_\theta = \rho d\rho d\varphi\hat{\theta}$	$\rho^2 d\rho sin\varphi d\varphi d\theta$

←

Divergence Theorem

compute ① $\underbrace{\oiint_{S_{cyl}} F.n_1 dS_1}_{LHS} = \underbrace{\iiint_V \rho\, dV}_{RHS} = \iiint_V \nabla.F\, dV$

equate ② LHS = RHS → compute F

wtf class 27

Stoke's Theorem

$\oint_c F.T ds = \iint_S J.n dS = \iint_S \nabla \times F \cdot n ds$

LHS ↓ RHS ↓

compute & equate → find F.