Complex no.s

How to polar ↔ xy

$|x+iy| = \sqrt{x^2+y^2} = |z|$

arg $x+iy = \theta$

$z = |z|(\cos\theta + i\sin\theta)$

$\sqrt{2}\left(\cos -\pi/4 + i\sin -\pi/4\right)$

$e^{i\theta}$

$1-i = \sqrt{2}$

arg

$\sqrt{2}(\cos\theta + i\sin\theta)$

$\sqrt{2}e^{-i\pi/4} = \sqrt{2}\cos\theta + i\sin\theta$

$\sqrt{2}\cos\theta = 1 \quad \cos\theta = 1/\sqrt{2}$

$\sqrt{2}\sin\theta = -1 \quad \sin\theta = -1/\sqrt{2}$

S A T C

$(-\pi \text{ to } \pi] \quad + 315° \quad -45$

180

$\boxed{-\pi/4}$

$+2k\pi$

$\angle a + 2k\pi$

log it

$\ln z = \log_e|z| + i\arg z$

$\ln_e \to$ complex

$\log_e \to$ real log

principal branch

$\mathrm{Ln}(z) = \log_e|z| + i\,\mathrm{Arg}(z)$

↳ log but principal

↳ principal argument

Complex powers

$z^a = w$

$z^a = e^{a\ln z}$

$e^{\ln_e w} = e^{a\ln z}$

$e^{\ln z^a}$

$w = e^{a\ln z}$

a

$z^a = e^{a\ln_e z}$

$z^n = r^n \angle n\theta$

i^i

$e^{i\ln_e i}$

$\ln i = \log_e|i| + i\arg z$

$i\theta$

$i = 1(\cos\theta + i\sin\theta)$

0, 1

$\pi/2$, 90°

$\ln i = i(\pi/2 + 2k\pi)$

$\boxed{e^{-(\pi/2 + 2k\pi)}}$

Principal Power

$e^{a\,\mathrm{Ln} z} = z^a$

trig

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

hyperbolic

$\cosh z = \dfrac{e^{z} + e^{-z}}{2}$ $\quad \sinh z = \dfrac{e^{z} - e^{-z}}{2}$

$\cos z = \cosh(iz)$ $\quad \sin z = \dfrac{\sinh(iz)}{i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad \dfrac{e^{iz} - e^{-iz}}{2i}$

$\dfrac{e^{z} + e^{-z}}{2}$ $\quad \dfrac{e^{z} - e^{-z}}{2}$

$\cos z = \cosh(iz)$

$\cosh(iz)$

$\sin z = \dfrac{\sinh(iz)}{i}$

$\sinh(iz)$

$\frac{d}{dz}\ln z = \frac{1}{z}$

$\ln(z_1 z_2) = \ln z_1 + \ln z_2$

$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

FODE

$y'' = 0$
$y = Cx + d$

LODE linear first ODE

$a_1(t)\ y' + a_0(t)\ y = g(x)$
↳ if $= 0$, homogenous

Standard form ⇒ y' coeff $= 1$

$y' + Py = Q$ ← multiply

IF : $e^{\int P\,dt}$
integrating factor

: product rule

$\text{IF} \cdot y = \int \text{IF} \cdot Q\, dt$

$\text{LHS} = e^{\int P\,dt}(y' + Py)$

$= e^{\int P\,dt} \frac{dy}{dt} + Py\, e^{\int P\,dt}$

$= \frac{d}{dt}\left(y \cdot e^{\int P\,dt}\right)$

$\int \frac{d}{dt}(y \cdot \text{IF}) = \int Q \cdot \text{IF}$

$y \cdot \text{IF} = \int Q \cdot \text{IF}\ dt$

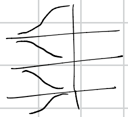
ODE $y'' = ky$ → linear
↓
second order

$y'' = ky^2$ → non linear

$y' + P(x)\ y = f(x)$

Autonomous DE

solution curves never intersect



Phase portrait

SODE

{HSOLODE
↳ RHS = 0

$Ay'' + By' + Cy = 0$

let $y = e^{rt}$
$y' = re^{rt}$
$y'' = r^2 e^{rt}$

$e^{rt}\underbrace{(Ar^2 + Br + C)}_{\text{Auxillary OR characteristic}} = 0$
↓
can't be 0

HODE

$$g(x) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_n(x)y' + a_1(x)y$$

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$$

Wronskian

$$W(f_1, f_2 \cdots f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & \ddots & & \\ \vdots & & \ddots & \\ f_1^{(n-1)} & & & f_n^{(n-1)} \end{vmatrix}$$

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$i'' + \frac{R}{L}i' + \frac{1}{CL}i = 0$$

existence and uniqueness
- → all coeff continuous
- → total sum continuous
- → initial coeff $\neq 0$

Homogenous? equation = 0

$y(x) = c_1 y_1(x) + c_2 y_2(x) \dots = 0$ only when all coeff 0 — linearly independent [finding non-trivial sol^n]

Wronskian $\neq 0$ — linearly independent

all functions are solution to the same HOLDE — linearly independent

$$i = C\frac{dV}{dt} \qquad V = -L\frac{di}{dt}$$

Reduction of Order

↳ put $y_2(x) = u(x)\,y_1(x)$

↳ find $y_2(x)$, $y_2'(x)$, $y_2''(x)$ & substitute

↳ find $u(x)$, add solutions = $y(x)$ general sol^n

↳ Wronskian if needed (linear dependence)

RLC $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$

- **overdamped** $D > 0$ $\quad c_1e^{m_1x} + c_2e^{m_2x} + \dots$
- **critically damped** $D = 0$ $\quad (c_1 + c_2x + c_3x^2 \dots c_kx^{k-1})\,e^{mx}$
- **underdamped** $D < 0$ $\quad e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

~~$e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x]$~~

Superposition

↳ addition of multiple solutions → also a solution

↳ homogenous? → trivial solution exist $y(x) = 0$

undetermined coeff

HONHDE

$y_{homo} + y_{particular}$

↓

assume y =

$r(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0$.)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x + b\sin\beta x$	$A\cos\beta x + B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0$.)
$ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a_0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Variation of Parameters

when no clear substitution (can be used for anything)

B R G O E

Key Points: *Lecture #11 Variation of Parameters*

1. The ***variation of parameters*** is another method which can be used to determine the ***particular solution to a nonhomogeneous DE***.
   - Together with the complementary solution we can form the general solution to the nonhomogeneous DE: $y(x) = y_c(x) + y_p(x)$
2. The variation of parameters is based upon taking the solutions to the ***associated homogeneous DE*** (the ***complementary solution***, e.g., $y_c = c_1y_1 + c_2y_2$) and "perturbing" them, to give the particular solution: $y_p = u_1y_1 + u_2y_2$
3. These "perturbing" functions are found by enforcing two auxiliary conditions for their derivatives:

$$\left.\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \underbrace{f(x)}_{\text{RHS}} \end{aligned}\right\} \text{condition}$$

   and solving this system, which gives the equations:

$$u_1(x) = \int u_1'(x)dx = -\int \frac{y_2 f}{W(y_1,y_2)}dx \qquad u_2(x) = \int u_2'(x)dx = \int \frac{y_1 f}{W(y_1,y_2)}dx$$

4. The variation of parameters can be applied to all linear differential equations, as long as we can find the complementary solution, i.e., the solution to $L(y) = 0$.
   - We have only discussed the case of 2[nd]-order DEs, but this can be extended to higher-order linear DEs

Laplace

Differential eqⁿ (separable, linear)

Laplace Transform $F(s) = \int_0^b e^{-st} f(t)\,dt$ $b \to \infty$

exists if

$f(t)$ → piecewise continuous

→ exponential order

$\lim_{s\to\infty} F(s) = 0$

s-domain	t-domain
$s = \sigma + j\omega$	time
frequency	

Inverse laplace transform

$$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} F(s)\,ds$$

#1 Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

#2 Time Differentiation

$$\mathcal{L}\{f^{(n)} t\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

① solving by partial fraction expansion

② cover up method: multiply both sides by $(s+p_1)$

evaluate both sides at $s = -p_1$

} distinct factors

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Repeated factors: $\frac{N(s)}{(s-p_1)^k} = \frac{A_k}{(s-p_1)^k} + \frac{A_{k-1}}{(s-p_1)^{k-1}} + \dots + \frac{A_2}{(s-p_1)^2} + \frac{A_1}{s-p_1}$

#3 s-shifting

$$F(s-a) = \mathcal{L}\{e^{at} f(t)\}$$

... $a = t_0$ in my notes

#4 t-shifting

$$e^{-t_0 s} F(s) = \mathcal{L}\{f(t-t_0)\, u(t-t_0)\}$$

OR heaviside

unit step function

$$u(t-a) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

activates the function t at t_0

$$\mathcal{L}\{u(t-t_0)\} = \frac{e^{-st_0}}{s} = F(s)$$

#5 Derivatives of transforms $\frac{d}{ds}F(s)$

$$t^n f(t) \Leftrightarrow (-1)^n F^{(n)}(s)$$

#6 Convolution

$$F(s)\,G(s) \Leftrightarrow \mathcal{L}\left\{\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right\} \Leftrightarrow (f * g)(t)$$

commutative $f(t) * g(t) = g(t) * f(t)$

associative $(f*g)*h = f*(g*h)$

distributivity $f*(g+h) = f*g + f*h$

#7 time integration

$$\int_0^t f(\tau)\,d\tau = \frac{F(s)}{s}$$

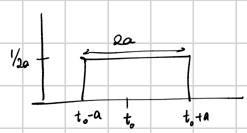
#8 periodic signals

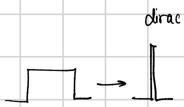
$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Dirac Delta Function

"sifting property"

$$\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)$$





$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

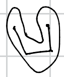
$$\mathcal{L}\{\delta(t)\} = 1$$

Complex

Sets and Functions

open set → interior points include

closed set → interior points, boundary points

connected → 2 points in set connected by finitely many straight line 

$w(z) = u(x,y) + j\,v(x,y)$ $z = x + iy$ $x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2}$

Limits, Continuity, Differentiation

continuity condition: $\lim_{z \to z_0} f(z) = f(z_0)$ rational function $f(z) = \frac{g(z)}{h(z)}$ $h(z) \neq 0$

derivative $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ limit exists? → differentiable

$\Delta z = \Delta x + i\Delta y$ →

Horizontal approach

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, $x \to 0$ $y = 0$

Vertical approach

$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, $x = 0$ $y \to 0$

Cauchy-Riemann Equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

→ complex differentiable (always true)
→ existence of $f'(z)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$e^{i\theta}$ → $x + iy$

$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x = v_y - iu_y$
$= u_x - iy_y = v_y + iv_x$

Analytic Functions

(Holomorphic) ↳ convergent power series

$f'(z)$ exists for points around z_0 too

↳ if not, it's singular

Harmonic Functions

$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$

$F(z) = u(x,y) + iv(x,y)$
analytic ↳ harmonic? ✓

Harmonic Conjugate → eg. (u, v are harmonic conjugates of each other)

must satisfy cauchy riemann

another, v → harmonic
→ analytic

in domain D
"harmonic conjugate of u"

Contour integration

$z(t) = x(t) + iy(t)$ parameterization

$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$ integration
↳ twice: real & imaginary

C doesn't have to be closed

$|g(z)| \leq M$ for all $z \in C$

max absolute value of $f(z)$ over C

$$\left| \int_C \underbrace{f(z)}_{g}\,dz \right| \leq ML$$

length of C

Properties

Linearity $\int_C [\alpha f(z) + \beta g(z)]\,dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz$

Path decomposition $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

Reversal of path orientation $= \int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

$|z - z_0| = \rho$

$$\int_C (z - z_0)^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq -1 \end{cases}$$

Line Integral

$\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j} = (P(x,y), Q(x,y))$

$W = \int_c \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_a^b P(x(t),y(t))\,x'(t)\,dt + \int_a^b Q(x(t),y(t))\,y'(t)\,dt = \int_c P\,dx + Q\,dy$

with scalar quantities

$\int_c f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$ $\qquad ds = |\vec{r}\,'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2}\,dt$

Circulation $= \oint u\,dx - v\,dy = \mathrm{Re}\{\oint_c f(z)\,dz\}$ $\qquad \vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ unit tangent vector $\qquad \oint f\cdot T\,ds$

Net flux $= \oint v\,dx + u\,dy = \mathrm{Im}\{\oint_c f(z)\,dz\}$ $\qquad \vec{N} = \frac{(y'(t), -x'(t))}{|\vec{r}\,'(t)|}$ unit normal vector $\qquad \oint f\cdot N\,ds$

$\oint_c f\cdot T\,ds + i\oint_c f\cdot N\,ds = \oint_c (u - iv)(dx + i\,dy) = \oint_c \overline{f(z)}\,dz$

$\vec{F}(x,y) = u(x,y)\hat{i} - v(x,y)\hat{j}$ $\qquad$ analytical complex function $\qquad f(z) = u(x,y) + i\,v(x,y)$ $\qquad$ zero divergence, zero curl $\qquad \nabla\cdot\vec{F} = 0$ and $\nabla\times\vec{F} = 0$

harmonic functions

Cauchy - Goursat

simple closed contour $\qquad$ closed contour $\qquad$ no holes → simply connected domain

Theorem: $\oint_C f(z)\,dz = 0$

principle of deformation of contours $\qquad \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$

multiply connected domain

$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$

analytic everywhere within C but exterior to all C_k's

independence of path

- Analytic
- simply connected domain
- → path independent

$\int_c f(z)\,dz = F(z_2) - F(z_1)$ $\qquad F(z) \rightarrow$ integral of $f(z)$

- analytic
- simply connected domain
- → antiderivative

$F'(z) = f(z)$

- analytic
- entire contour of integration
- → integrate

Ln(z) analytic
domain doesn't include branch cut (non positive real axis)
hence domain ⇒ $|z| > 0$ $\qquad -\pi < \mathrm{Arg}(z) < \pi$

eg. domain: $\frac{d}{dz}\mathrm{Ln}(z) = \frac{1}{z}$

Cauchy

$f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0}\,dz$ $\qquad$ "Cauchy Integral Formula" $\qquad$ first order singularities

→ infinitely differentiable, derivatives analytic within domain

if: $f(z) \rightarrow$ analytic on and within contour C

$f^{(n)}(z_0) = \frac{n!}{2\pi j}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$ $\qquad$ "Extended Cauchy Integral Formula" $\qquad$ higher order singularities

Liouville

- bounded
- entire
- → constant

for P(z) being a non-constant polynomial, then P(z) = 0 must have at least one root

ML inequality

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} = \frac{n!M}{r^n}$$

Sequences and Series

$\{ z_n = x_n + jy_n \} \to L$
if only $x_n \to Re(L)$, $y_n \to Im(L)$ $[n \to \infty]$

Geometric series: $S_n = \frac{a(1-z^n)}{1-z}$ $\quad a, az, az^2 \dots$

$\frac{a}{1-z}$ when z is small, n is large

$$\sum z^n = \frac{1}{1-z}$$

$$\sum az^n = \frac{a}{1-z}$$

Tests of convergence

n^{th} term test

Ratio Test $L = \lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right|$
$\lim_{n\to\infty} z_n = 0$

Root Test $L = \lim_{n\to\infty} \sqrt[n]{|z_n|}$
$\lim_{n\to\infty} z_n = 0$

Series converges absolutely if $L<1$
... diverges ... if $L>1$
test inconclusive if $L=1$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \begin{cases} L \neq 0 & R = 1/L \\ = 0 & R = \infty \\ = \infty & R = 0 \end{cases}$

Series divergence
$\lim_{n\to\infty} z_n \neq 0$

Radius of Convergence

of $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ $\to$ root OR radius test

if $R=\infty$ converges to L
if $R=0$ converges at centre $z=z_0$
if $0<R<\infty$ converges within circle of radius R about point $z=z_0$

Taylor Series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(z_0)}{k!} (z-z_0)^k$$

$|z-z_0| < R$

when $z_0 = 0 \to$ Maclaurin

$e^w = 1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \dots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$

Substitution method: $f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ $\quad g(z) = \frac{1}{1+z} = \sum_{k=0}^{\infty} (-z)^k = \sum_{k=0}^{\infty} (-1)^k z^k$

Geometric series: $z_0 = 1$ $\quad$ eg. $\frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{-1/2}{1-\frac{z-1}{2}} = -\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z-1}{2}\right)^k$

Laurent Series

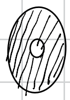
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

$a_k = \frac{1}{2\pi j} \oint_C \frac{f(w)}{(w-z_0)^{k+1}} dw$ $\quad k = 0, \pm 1, \pm 2 \dots$

$a_{(-1)} = \frac{1}{2\pi j} \oint f(w)\, dz \to$ residue

$\hookrightarrow \oint f(z)\, dz = 2\pi i\, a_{(-1)}$

principal part: $(z-z_0)^{-ve}$ $\quad r < |z-z_0|$ converge
analytic part: $(z-z_0)^{+ve}$ $\quad |z-z_0| < R$ converge



Zeros and Singularities

$\downarrow$ poles

isolated singularity
- removable $\to$ 0 principal terms
- simple pole $\to$ 1 principal term
- pole $\to$ finite principal term
- essential singularity $\to \infty$ principal terms

Q) Compute $\oint_{|z|=2} z^5 \cos\left(\frac{1}{z}\right) dz$

$f(z) = z^5 - \frac{1}{2}z^3 + \frac{z}{4!} - \frac{1}{6!\,z} + \dots$

$\hookrightarrow -\frac{1}{6!} \to$ residue

$\oint f(z)\, dz = 2\pi i \left(\frac{-1}{6!}\right) = \frac{-2\pi i}{6!}$

"I'm just a simple pole
in a complex plane"

order = number of -ve terms in principal terms

if $f(z) = \frac{g(z)}{h(z)}$ $\quad$ zeros $\to$ highest power / order / zeros = m; poles $\to n$

$\downarrow$
has a pole of order $(n-m)$ if $n>m$
has a zero of order $(m-n)$ if $m>n$
removable singularity $(n=m)$

Residue

Laurent series $k=-1$ term

$$\oint_C f(z)dz = \oint_C \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \, dz = 2\pi j \, a_{(-1)} = 2\pi j \, Res(f(z) \text{ at } z_0)$$

Simple pole: $Res = \lim_{z \to z_0} (z-z_0) f(z)$

$$Res(f(z), z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z)\right)$$

n = order of pole

lim f(z)

Sum of residues of function at that singularity

$$\oint_C f(z)dz = 2\pi j \sum_{k=1}^{n} Res(f(z), z_k)$$

Real integrals

$z = e^{j\theta}$

$d\theta = \frac{dz}{jz}$

$\cos(n\theta) = \frac{e^{jn\theta} + e^{-jn\theta}}{2}$

$\sin(n\theta) = \frac{e^{jn\theta} - e^{-jn\theta}}{2j}$

Principal Value: $\int_{-\infty}^{\infty} \to \lim_{R\to\infty} \to \int_{-R}^{R} \to PV$

if $\int_{-\infty}^{\infty} f(x)dx$ convergent, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

$PV \int_{-\infty}^{\infty} f(x)\,dx = 2\pi j \sum Res(f(z))$

Jordan's Lemma

$z = Re^{i\theta}$

$\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$\alpha > 0$	$\lim_{R\to\infty} \int_{C_R^+} f(z) e^{i\alpha z} dz = 0$	counter clockwise	upper half circle	$0 \le \theta \le \pi$
$\alpha < 0$	$\lim_{R\to\infty} \int_{C_R^-} f(z) e^{i\alpha z} dz = 0$	clockwise	lower half circle	$0 \le \theta \le -\pi$

$$\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x) dx + i \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx$$

$$PV \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \pm 2\pi i \sum_{i=1}^{n} Res(f(z) e^{i\alpha z})$$

Residue theorem

Indented Contour

$$\lim_{r\to 0} \int_{C_r} f(z)dz = 2\pi i \left[\frac{\alpha}{2\pi} Res(f(z), z_0) \right]$$

simple pole on real axis

$$P.V. \int_{-\infty}^{\infty} f(x)dx = 2\pi i \left[\sum_{\substack{\text{all poles} \\ \text{within } C}} Res(f(z), z_k) \right] + \pi i \left[\sum_{\substack{\text{all simple poles on} \\ \text{real axis}}} Res(f(z), z_k) \right]$$

assuming $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

Inverse Laplace Transform

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds = \sum_{k=1}^{n} Res(F(s)e^{st}, s_k)$$